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NAVAL POSTGRADUATE SCHOOL

Monterey, California

A MODEL FOR
THE DEFENSE OF A MINE FIELD

by
P. A. Jacobs

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This paper presents and analyzes a simple stochastic model for defense against an attacking force of tanks; the defense is made up of a mine field and a single defending tank. The approach of the paper makes use of classical applied probability notions and techniques and explicit algebraic solutions are derived that can easily yield numerical results and hence interpretable insights into the value of various tactics.

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A MODEL FOR THE DEFENSE OF A MINE FIELD

by

P. A. Jacobs

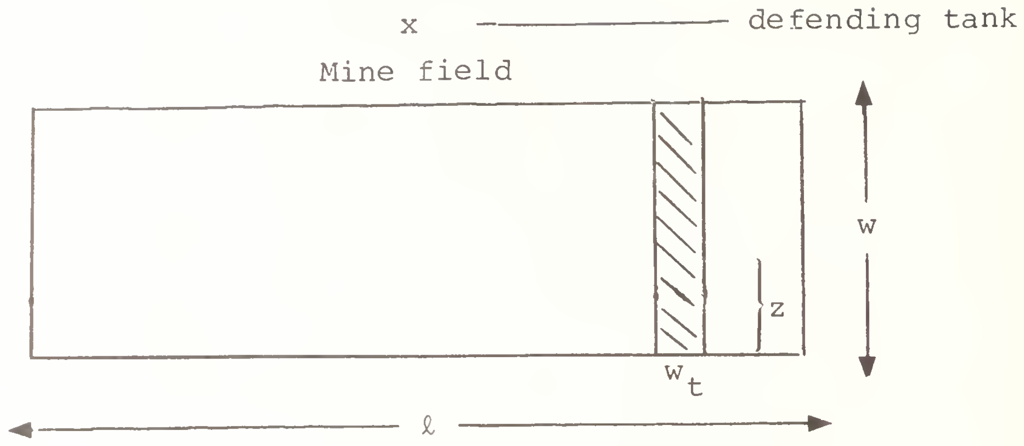
Introduction and Summary

This paper presents and analyzes a simple model for defense against an attacking force of tanks; the defense is made up of a mine field and a single defending tank. Extensions that include many defending tanks are possible, but the algebra becomes difficult.

The approach of the paper makes use of classical applied probability notions and techniques, and explicit algebraic solutions are derived that can easily yield numerical results, and hence interpretable insights into the value of various tactics. The relative simplicity of the solutions should be attractive and useful as a supplement to much more realistic, but complex, simulations, and to time consuming and expensive war games. It is even possible that the present simple engagement analysis--and others like it--may be incorporated into more complex war games as important modular components.

Mine Field Assumptions

The field is of length ℓ and width w.



The position of mines in the field form a spatially homogeneous Poisson process with rate r_m; that is, the number of mines in disjoint subsets of the field are independent random variables and the distribution of the number of mines in a subset A of the field is Poisson with mean $r_m|A|$ where $|A|$ denotes the area of A (cf. Feller [1971]). Assume tanks are of width w_t and they travel through the field parallel to the w edge of the field. (The cross-hatched area of the diagram represents a typical potential path.) The mines are "invisible" so there is no evasive action taken by the tanks. By the Poisson assumption, the probability that a tank gets z units into the field without hitting a mine equals $\exp(-r_m w_t z) \equiv e^{-\mu z}$.

Defending Tank Assumptions

The defending tank is located on the far side of the field, and can fire on offensive tanks crossing the field. The sweep rate of a defending tank over the field is a (mi/sec). It takes $\frac{\ell}{a}$ sec to sweep the entire field. The probability of detecting an existing tank in the field during a sweep is p. Tanks travel at c mi per sec. Hence it takes a tank $\frac{w}{c}$ sec to cross the field. During this crossing time a defending tank that continually sweeps has approximately $\frac{w}{c}\frac{a}{\ell}$ chances to detect the offensive tank. In this situation the probability of detecting the tank before it crosses the field is approximately

$$1 - (1-p)^{\frac{w}{c}\frac{a}{\ell}} = 1 - \exp\{- w \frac{a}{c\ell} [-\ell n(1-p)]\}$$

$$\equiv 1 - \exp\{-\gamma w\}.$$

An analogous argument can be used to argue that the probability that a defensive tank that continually sweeps detects an offensive tank before it is x units into the field is $1 - \exp\{-\gamma x\}$. For simplicity we will assume that when an offensive tank is detected it is killed.

If there are k tanks traveling on disjoint paths in the field the probability of detecting at least one of the tanks before they go a distance x units into the field is approximately

$$1 - (1-p)^{k\frac{x}{c}\frac{a}{\ell}} \equiv 1 - \exp\{-k\gamma x\}.$$

Procedure of Defending Tank when an Offensive Tank is Killed

If an offensive tank is killed a distance z into the field by either a mine or the defensive tank, the defensive tank has the option of doing a limited sweep about z in the hope of detecting other tanks which are with the killed tank in a convoy; we will assume that the defensive tank sweeps a distance d miles about the place z for n times. This limited sweep takes $\frac{d}{a}n$ sec. During this time an offensive tank can go $\tau = c\frac{d}{a}n$ miles. During each sweep there is a probability α of detecting another offensive tank if it is in the limited sweep area. Given that an offensive tank is in the limited sweep area, the probability of detecting it before it goes an additional x units into the field is approximately

$$1 - (1-\alpha)^{\frac{a}{cd}x} \equiv 1 - \exp\{-\delta x\} \qquad \text{for} \quad x \leq \tau .$$

Convoys Assumption

We will assume that the probability of detecting an offensive tank in a convoy before it goes a distance z into the field is $1 - \exp\{-\eta z\}$ where $\eta \geq \gamma$. The probabilities of detecting individual tanks in a convoy are independent.

We will assume that tanks in a convoy follow the same path through the field.

As a simplifying assumption we will assume that all tanks in a convoy start at the same time. Different starting times can be modelled but seem to make calculations more difficult. As a result, if there are two tanks in a convoy we assume that the probability of detecting at least one of them before the convoy goes a distance z is $1 - \exp\{-2\eta z\}$. If a detection occurs, we assume that only one tank is detected (and therefore killed).

Derivation of Results for a Simple Model

To illustrate the calculations involved we will assume there are three offensive tanks and one defensive tank. We will assume that the defensive tank never goes down. Let N be the number of defensive tanks that get through the mine field. We will assume the defense wins if $N \leq 1$.

Scenario I. The three offensive tanks go through the field on disjoint paths. The paths are far enough apart so that a limited sweep about the area of one will not detect tanks on the other paths. The offensive tanks all start through the field at the same time.

If there is no defensive tank, then the only way an offensive tank can be killed is by hitting a mine. Since the tanks go through the field on disjoint paths, the probability none of them hits a mine before a distance of z into the field is $(e^{-\mu z})^3$. The density function of the position of the first mine to be hit is $3\mu e^{-3\mu z}$, $z \leq w$. Once the first mine is hit two tanks are still going through the field. By the lack of memory property for the exponential distribution the distribution of the distance until a mine is hit is a truncated exponential with rate 2μ. Hence,

$$P\{N \leq 1\} = \int_0^w 3\mu e^{-3\mu z} [1 - e^{-2\mu(w-z)}]dz$$

$$= [1 - e^{-3\mu w}] - e^{-2\mu w} \frac{3\mu}{\mu} [1 - e^{-\mu w}]$$

$$= 1 - e^{-3\mu w} - 3e^{-2\mu w} [1 - e^{-\mu w}]$$

If there is a defensive tank, then the distance until the first tank is killed (either by a mine or the defensive tank) has a density function $3(\mu + \gamma)e^{-3(\mu+\gamma)z}$ for $z \leq w$. Once an offensive tank is detected at a distance z into the field, the defensive tank does a limited sweep about the detection area. By assumption the defending tank does not detect the other offensive tanks during this limited sweep. During a limited sweep the offensive tanks can go an additional distance of τ units into the field unless they hit a mine.

By the lack of memory property of the exponential the density function of the distance before an offensive tank hits a mine is $2\mu e^{-2\mu y}$, $0 \leq y < \min(\tau, w-z)$. If neither offensive tank hits a mine during the limited sweep the density function of the distance until an offensive tank is detected is $2(\mu+\gamma)e^{-2(\mu+\gamma)y}$, $0 \leq y \leq w - \tau - z$. Hence

$$
\begin{aligned}
P\{N \leq 1\} &= \int_{w-\tau}^{w} 3(\gamma+\mu)\, e^{-3(\gamma+\mu)z}\, [1 - e^{-2\mu(w-z)}] \\
&\quad + \int_{0}^{w-\tau} 3(\gamma+\mu)\, e^{-3(\gamma+\mu)z}\, [1 - e^{-2\mu\tau}] \\
&\quad + \int_{0}^{w-\tau} 3(\gamma+\mu)\, e^{-3(\gamma+\mu)z}\, e^{-2\mu\tau}\, [1 - e^{-2(\gamma+\mu)(w-\tau-z)}] \\
&= 1 - e^{-3(\gamma+\mu)w} \\
&\quad - e^{-2\mu w} \frac{3(\gamma+\mu)}{3\gamma+\mu} [e^{-3(\gamma+\mu)(w-\tau)} - e^{-(3\gamma+\mu)w}] \\
&\quad - e^{-2\gamma(w-\tau)}\, e^{-2\mu w}\, 3[1 - e^{-(\gamma+\mu)(w-\tau)}] .
\end{aligned}
$$

<u>Scenario II.</u> All the tanks go through the field in a convoy.

a) If there is no defending tank, then

$$P\{N \leq 1\} = 1 - e^{-\mu w} - \mu w e^{-\mu w} .$$

which is the probability of there being two or more mines in a rectangle of width w_t and length w.

b) If there is the defending tank, then the distance the offensive tanks go until one is detected has a truncated exponential distribution with rate $3\eta + \mu$. Once an offensive tank is detected the defensive tank begins a limited sweep. During the limited sweep the distribution of the distance the remaining tanks can go before being detected is truncated exponential with rate $2\delta + \mu$. If no tank is detected during the limited sweep and the offensive tanks are still in the mine field the distribution of the distance the offensive tanks go before being detected is a truncated exponential with rate $2\eta + \mu$. Hence

$$P\{N \leq 1\} = \int_{w-\tau}^{w} (3\eta+\mu)\, e^{-(3\eta+\mu)z}[1 - e^{-(2\delta+\mu)(w-z)}]dz$$

$$+ \int_{0}^{w-\tau} (3\eta+\mu)\, e^{-(3\eta+\mu)z}\, [1 - e^{-(2\delta+\mu)\tau}]\, dz$$

$$+ \int_{0}^{w-\tau} (3\eta+\mu)\, e^{-(3\eta+\mu)z}\, e^{-(2\delta+\mu)\tau}\, [1 - e^{-(2\eta+\mu)(w-\tau-z)}]dz$$

$$= 1 - e^{-(3\eta+\mu)w}$$

$$-e^{-(2\delta+\mu)w}\, \frac{3\eta+\mu}{3\eta-2\delta}\, [e^{-(3\eta-2\delta)(w-\tau)} - e^{-(3\eta-2\delta)(w)}]$$

$$- e^{-(2\eta+\mu)w}\, \frac{3\eta+\mu}{\eta}\, [1 - e^{-\eta(w-\tau)}]\, e^{-2(\delta-\eta)\tau}$$

if $3\eta \neq 2\delta$. If $3\eta = 2\delta$, then

$$P\{N \leq 1\} = 1 - e^{-(3\eta+\mu)w}$$

$$-e^{-(2\delta+\mu)w}(3\eta+\mu) - e^{-(2\eta+\mu)w}\,\frac{3\eta+\mu}{\eta}\,[1-e^{-\eta(w-\tau)}]e^{-2(\delta-\eta)\tau} .$$

<u>Scenario III</u>. Two convoys with two tanks in one and one tank in the other. The convoys are far enough apart so that at limited sweep about the area of one will not detect the other. The convoys start at the same time.

If there is no defensive tank, then

$$P\{N \leq 1\} = 1 - e^{-2\mu w} - \mu w e^{-2\mu w} - e^{-\mu w}[1 - e^{-\mu w}]$$

which is the probability of there being at least two mines in the path of the convoy of size 2 or at least 1 mine in each of the two paths.

If there is a defensive tank, then the distribution of the distance until the first detection is a truncated exponential with rate $2\eta + \gamma + 2\mu$. If the tank in the convoy of size one is detected, then during a limited sweep, tanks in the other convoy can only be detected by encountering a mine; hence, by the lack of memory property of the exponential, during the limited sweep the density function of the distance to detection is a truncated exponential with rate μ; if during the limited sweep a tank is not detected, the density function of the distance after the limited sweep until a tank is detected is a truncated exponential with rate $2\eta + \mu$. If one of the tanks in the convoy of size two is detected, then during the limited

sweep the other tank in the convoy may be detected by the defensive tank or a mine while the tank in the other convoy can only be detected by a mine; hence, during the limited sweep the density function of the distance until an offensive tank is detected is a truncated exponential with rate $2\mu + \delta$; if no tank is detected during the limited sweep, the density function of the distance after the limited sweep until a tank is detected is a truncated exponential with rate $2(\mu + \gamma)$.

$P\{N \leq 1\}$

$$= \int_{w-\tau}^{w} dy\, e^{-(2\eta+\gamma+2\mu)y} \{(\gamma+\mu)[1-e^{-\mu(w-y)}]+(2\eta+\mu)[1-e^{-(\delta+2\mu)(w-y)}]\}$$

$$+ \int_{0}^{w-\tau} dy[2\eta+\mu]e^{-(2\eta+\gamma+2\mu)y} \{[1-e^{-(\delta+2\mu)\tau}] + e^{-(\delta+2\mu)\tau} [1-e^{-2(\gamma+\mu)(w-\tau-y)}]\}$$

$$+ \int_{0}^{w-\tau} dy[\gamma+\mu]e^{-(2\eta+\gamma+2\mu)y} \{[1-e^{-\mu\tau}]+e^{-\mu\tau}[1-e^{-(2\eta+\mu)(w-\tau-y)}]\}$$

$$= [1 - e^{-(2\eta+\gamma+2\mu)w}]$$

$$- e^{-\mu w} \frac{\gamma+\mu}{2\eta+\gamma+\mu} [e^{-(\gamma+2\eta+\mu)(w-\tau)} - e^{-(\gamma+2\eta+\mu)w}]$$

$$- e^{-(\delta+2\mu)w} \frac{2\eta+\mu}{2\eta+\gamma-\delta} [e^{-(2\eta+\gamma-\delta)(w-\tau)} - e^{-(2\eta+\gamma-\delta)w}]$$

$$- e^{-2\eta(w-\tau)} e^{-\mu w} [1 - e^{-(\gamma+\mu)(w-\tau)}]$$

$$- \frac{2\eta+\mu}{2\eta-\gamma} e^{-2\gamma(w-\tau)} e^{-2\mu w} e^{-\delta\tau} [1 - e^{-(2\eta-\gamma)(w-\tau)}]$$

if $2\eta + \gamma - \delta > 0$ and $2\eta - \gamma > 0$. If $2\eta + \gamma - \delta = 0$, then the third term becomes

$$(2\eta + \mu)\tau \, e^{-(\delta+2\mu)w} .$$

If $2\eta - \gamma = 0$, then the last term becomes

$$e^{-2\gamma(w-\tau)} \, e^{-2\mu w} \, e^{-\delta\tau} \, (2\eta + \mu)(w-\tau) .$$

Some Numerical Results for the Simple Model

We will now present some numerical results for the three scenarios above for some different parameter values.

Case I. Parameters: $w = 4$, $\mu = .1$, $\gamma = .1$, $\tau = 0$.

In this case the defending tank does not sweep a more limited area when an offensive tank is detected. The parameter η, the rate of detection of a tank in a convoy, is allowed to vary.

η / Scenario	$P\{N \leq 1\}$			
	.1	.2	.3	∞
I	.575	.575	.575	.575
II	.4	.68	.84	1
III	.5	.63	.69	1

Case II. Parameters: $w = 4$, $\mu = .1$, $\gamma = .1$, $\eta = .1$, $\tau = 1$. In this case δ, the rate of detection during a limited sweep, is allowed to vary

Scenario \ δ	$P\{N \leq 1\}$			
	.1	.2	.3	∞
I	.575	.575	.575	.575
II	.4	.504	.58	.8
III	.45	.45	.57	.67

The best strategy for the offensive is to choose that one for which the $P\{N \leq 1\}$ is the smallest. For Case I, $\eta = .1$, this strategy is Scenario II--send all tanks in one convoy; if $\eta \geq .2$ the best strategy is to send all tanks in separately. For Case II, $\delta = .1$, the best strategy is to send all the tanks in one convoy; if $\delta = .2$ or $.3$, then the best strategy is to use two convoys; if $\delta = \infty$, the best strategy is for each tank to go in separately.

Conclusions. The above model is very simple. However, it is complicated enough to show that for the offense different strategies are better under different scenarios. Other scenarios that one might want to include in the model are several defensive tanks; the firing of offensive tanks at defensive tanks; and offensive tanks entering the mine field at different times. These situations can all be modelled at the cost of a more complicated model and of course more complicated calculations.

REFERENCES

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